The Japanese Theorem

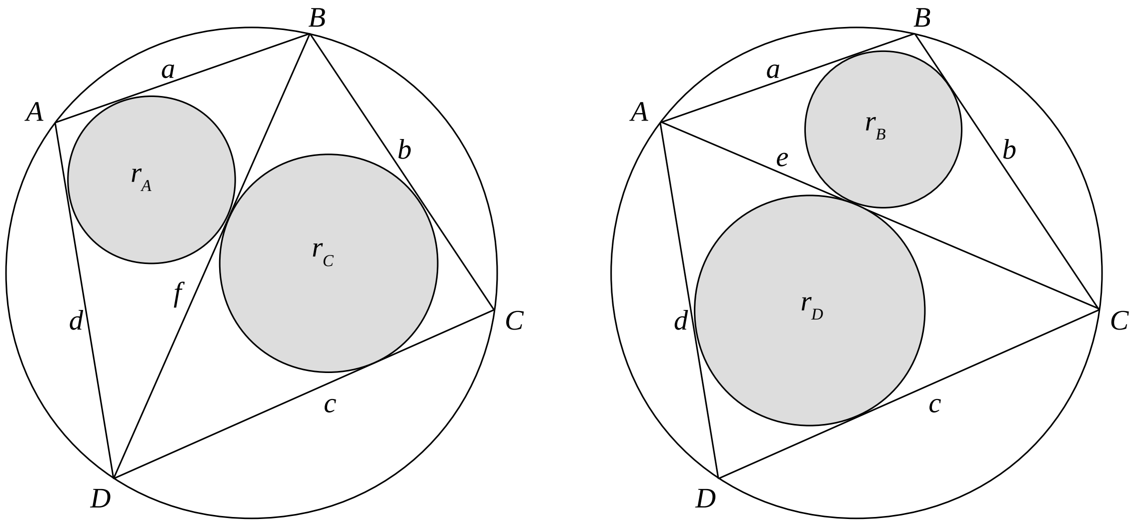
This result was known to the Japanese mathematicians during the period of isolation known as the Edo period. The proof requires only the most elementary geometry, but is not easy.

Statement

When a convex cyclic quadrilateral is divided by a diagonal into two triangles, the sum of the radii of the incircles of the triangles is independent of which diagonal is chosen.

So, in the example below, we have the result:

$$r_A + r_C = r_B + r_D$$



Proof

To prove this, we can use both of Ptolemy's theorems and some ingenious algebraic manipulation.

With the notation given in the diagrams, by Ptolemy's first theorem, we have

$$ac+bd = ef \quad \text{...(1)}$$

and by his second theorem:

$$\frac{e}{f} = \frac{ad+bc}{ab+cd}$$

which we can more usefully write as:

$$e(ab+cd) = f(ad+bc) \quad \text{..(2)}$$

If R is the radius of the circumscribing circle, then we have the four relations:

$$r_A = \frac{adf}{2R(a+d+f)} \quad ; \quad r_C = \frac{bcf}{2R(b+c+f)} \quad \text{...............................(3)}$$

$$r_B = \frac{abe}{2R(a+b+e)} \quad ; \quad r_D = \frac{cde}{2R(c+d+e)} \quad \text{...............................(4)}$$

From (3), by multiplying by $2R$ and adding, we get:

$$2R(r_A+r_C) = \frac{adf}{a+d+f} + \frac{bcf}{b+c+f}$$

Putting the right hand side onto a common denominator:

$$2R(r_A+r_C) = \frac{adf(b+c+f) + bcf(a+d+f)}{(a+d+f)(b+c+f)} = \frac{P}{Q} \quad \text{.......................(5)}$$

where P and Q represent the numerator and denominator expressions respectively.

Similarly from (4) we have:

$$2R(r_B+r_D) = \frac{abe(c+d+e) + cde(a+b+e)}{(a+b+e)(c+d+e)} = \frac{U}{V} \quad \text{.......................(6)}$$

The strategy is now to show that $eP = fU$, and $eQ = fV$, and hence the ratios are equal.

First we consider the numerators.

From (5), factoring out f and expanding the brackets:

$$P = f\{ad(b+c) + bc(a+d) + f(ad+bc)\}$$

Now, we have by expanding and re-arranging:

$$ad(b+c) + bc(a+d) = abd+acd+abc+bcd = ab(c+d) + cd(a+b)$$

Also, from (2), $\quad f(ad+bc) = e(ab+cd)$

and so by substituting for these last two expressions:

$$P = f\{ab(c+d) + cd(a+b) + e(ab+cd)\}$$

Re-arranging terms and extracting common factors again:

$$P = f\{ab(c+d+e) + cd(a+b+e)\}$$

Multiplying by e within the braces, and comparing with (6) we get:

$$eP = f\{abe(c+d+e) + cde(a+b+e)\} = fU \quad \text{...........................(7)}$$

Secondly, we consider the denominators.

Expanding

$$Q = (a+d+f)(b+c+f) = (a+d)(b+c) + f(a+b+c+d) + f^2$$

Multiplying through by e and expanding the first pair of brackets:

$$eQ = ef(a+b+c+d) + e(ab+ac+bd+cd) + ef^2$$

Looking at the middle term, we know from (1) that $ac+bd = ef$ and also that from (2), $e(ab+cd) = f(ad+bc)$ and so

$$e(ac+bd) + e(ac+bd) = e^2f + f(ad+bc)$$

In the last term, we can again use (1)

$$ef^2 = f(ef) = f(ac+bd)$$

Substituting these last two values in the expression for eQ, we get:

$$eQ = fe(a+b+c+d) + e^2 f + f(ad+bc) + f(ac+bd)$$

we can now extract the factor f and re-arrange the terms:

$$eQ = f\{e(a+b+c+d) + (ac+ad+bc+bd) + e^2\}$$

Collecting terms and comparing with (6)

$$eQ = f\{(a+b)(c+d) + e(a+b+c+d) + e^2\} = f(a+b+e)(c+d+e) = fV$$

Hence we can write:

$$\frac{P}{Q} = \frac{eP}{eQ} = \frac{fU}{fV} = \frac{U}{V}$$

Going back to (5) and (6), this means that

$$2R(r_A + r_C) = 2R(r_B + r_D)$$

and finally by dividing by $2R$:

$$r_A + r_C = r_B + r_D \quad \text{...} \blacksquare$$

Corollary

In any convex cyclic polygon, which has been divided into triangles by drawing diagonals between vertices, the sum of the radii of the inscribed circles within each of the triangles, is independent of the triangulation.

This will be apparent when one considers continually replacing suitable diagonals in quadrilaterals one by one until one triangulation is transformed into any other.

Alternative proof

The result can also be obtained as a direct consequence of a little known theorem by Carnot. This is not Sadi Carnot of thermodynamic and heat engine renown, but his father Lazare Carnot.

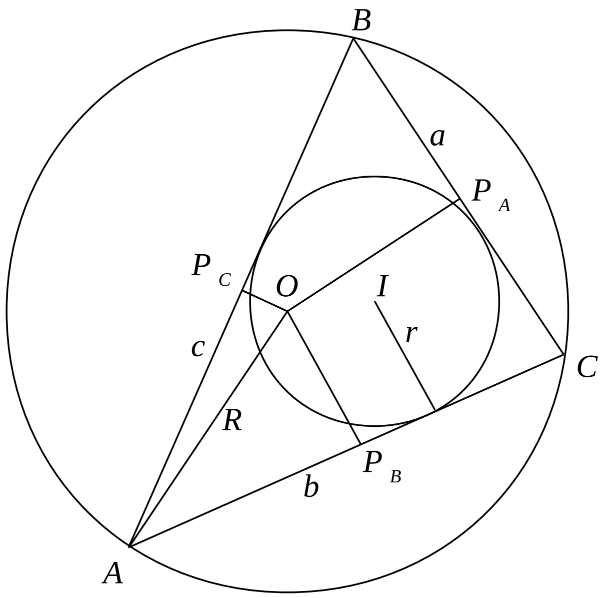
Carnot's Theorem

The sum of the perpendicular distances (suitably signed) from the circumcentre to the sides of a triangle is equal to the sum of the circumradius and the inradius.

That is in the diagram, where O is the circumcentre, and P_A, P_B, and P_C are the feet of the perpendiculars to the sides of the triangle opposite A, B and C respectively.

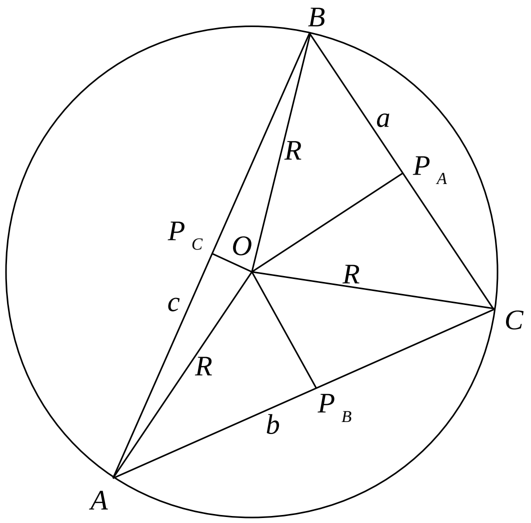
$$OP_A + OP_B + OP_C = R + r$$

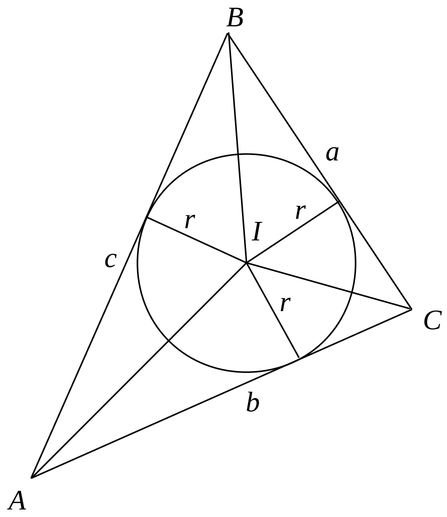
In the diagram, the circumcentre lies inside the triangle, in which case all distances are positive. But if one of the angles was obtuse (say at C), then O would lie outside. In such a case, the distance OP_C would be taken to be negative in length.



We will prove here the case of an acute angled triangle, and leave the reader to go through the other case to verify the signs work out right.

First, we compute the area of the triangle ABC in two different ways – by adding up the areas of three triangles with apices at the circumcentre, O, or the incentre, I.





In the left diagram, we have that the area of triangle ABC is the sum of the areas of the triangles BOC, COA and AOB. If we take the bases of these triangles as the sides of ABC, then the heights of each in turn are the lines OP_A, OP_B and OP_C respectively. Summing the areas, and doubling gives:

$$2T \;=\; aOP_A \;+\; bOP_B \;+\; cOP_C \quad \text{.......................................(1)}$$

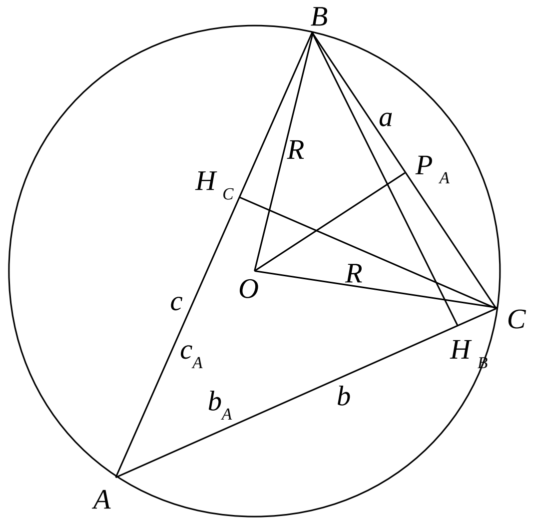
In the right hand diagram, the triangles *BIC*, *CIA* and *AIB* each have bases on the sides of triangle *ABC* and all have height *r*, the inradius. Hence summing these to get the area of the larger triangle and doubling, as before:

$$2T \;=\; ar \;+\; br \;+\; cr \;=\; r(a+b+c) \quad \text{...................................(2)}$$

equating (1) and (2):

$$aOP_A \;+\; bOP_B \;+\; cOP_C \;=\; r(a+b+c) \quad \text{.................................(3)}$$

Next, we put in a few more lines to help. Drop perpendiculars from *B* and *C* onto sides *b* and *c* to meet them at H_B and H_C respectively. Label the lengths AH_B and AH_C as b_A and c_A respectively, and other lengths analogously as required.



Since *O* is the centre of the circumcircle, angle *BOC* is twice angle *BAC*. Since angle OP_AB is a right angle, then angle BOP_A is equal to angle *BAC*. This means that the three triangles

AH_CC, AH_BB and OP_AB

are all similar to one another. Comparing the ratio of corresponding sides in each of these gives:

$$\frac{c_A}{b} \;=\; \frac{b_A}{c} \;=\; \frac{OP_A}{R}$$

We can continue:

$$\frac{OP_A}{R} \;=\; \frac{c_A+b_A}{b+c}$$

or more readily useful:

$$OP_A(b+c) \;=\; R(c_A+b_A) \quad \text{...(4)}$$

In a similar way, by looking at each of the angles *B* and *C* in turn, we can also derive:

$$OP_B(a+c) \;=\; R(a_B+c_B) \quad \text{...(5)}$$

and

$$OP_C(a+b) \;=\; R(a_C+b_C) \quad \text{...(6)}$$

Now we can add all the equations (3), (4), (5) and (6), and find:

$$OP_A(a+b+c)+OP_B(a+b+c)+OP_C(a+b+c) \;=\; R(a_B+a_C+b_A+b_C+c_A+c_B) \;+\; r(a+b+c)$$

Finally, noting that $a_B+a_C=a$ etc., and dividing by $(a+b+c)$:

$$OP_A \;+\; OP_B \;+\; OP_C \;=\; R \;+\; r \quad \text{..}\blacksquare$$

We use the sign convention that when a length OP_X lies partly or wholly inside the triangle, then it is positive, but if it lies wholly outside the triangle, then it is negative. This result is left as an exercise to verify – by repeating the calculations and noting that in such a case one of the triangles contributing to the total area must be subtracted from the sum of the other two, hence providing the motivation for the convention.

The Japanese Theorem

We can use this result to complete a proof of the Japanese theorem.

In a triangulation of an arbitrary cyclic *n*-gon, the lines perpendicular from the centre of the circumscribing circle and one of the diagonals of the polygon will occur twice when we add up the expressions for the radii of the inscribed circles of each triangle. One in each case will be negative, and one positive. Hence these will all cancel out.

The lines drawn to the sides of the polygon will occur just once and will always contribute the same signed value.

The circumradius of each of the triangles is the same, viz., R, and the number of the triangles is also always the same whatever the triangulation, viz n-2. Suppose that S is the sum of the (signed) perpendiculars from O to the sides of the polygon, and let the sum of the radii of the incircles be σ.

Then summing up the results of Carnot's theorem for each triangle will give

$$S \;-\; R(n-2) \;=\; \sigma$$

which shows that the sum does not depend on the triangulation, as required............................■

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