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A Theorem on Orthology Centers

Eric Danneels and Nikolaos Dergiades

Abstract. We prove that if two triangles are orthologic, their orthology centers have the same barycentric coordinates with respect to the two triangles. For a point P with cevian triangle $A'B'C'$, we also study the orthology centers of the triangle of circumcenters of $PB'C'$, $PC'A'$, and $PA'B'$.

1. The barycentric coordinates of orthology centers

Let $A'B'C'$ be the cevian triangle of P with respect to a given triangle ABC. Denote by O_a, O_b, O_c the circumcenters of triangles $PB'C'$, $PC'A'$, $PA'B'$ respectively. Since O_bO_c, O_cO_a, and O_aO_b are perpendicular to AP, BP, CP, the triangles $O_aO_bO_c$ and ABC are orthologic at P. It follows that the perpendiculars from O_a, O_b, O_c to the sidelines BC, CA, AB are concurrent at a point Q. See Figure 1. We noted that the barycentric coordinates of Q with respect to triangle $O_aO_bO_c$ are the same as those of P with respect to triangle ABC. Alexey A. Zaslasky [7] pointed out that our original proof [3] generalizes to an arbitrary pair of orthologic triangles.

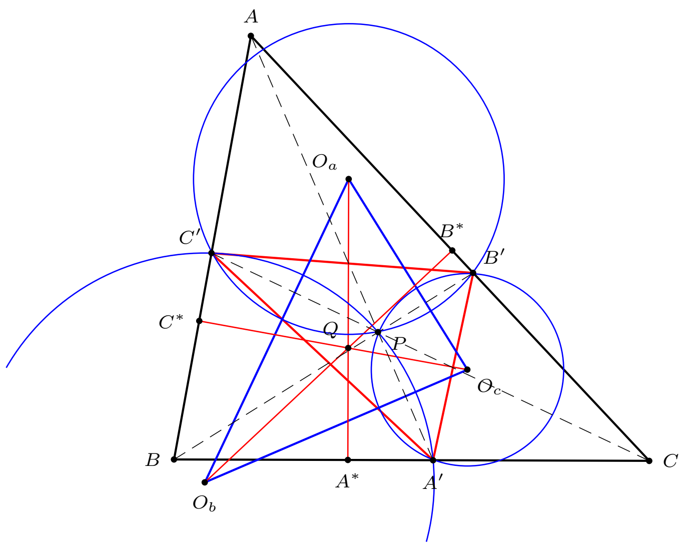

Figure 1

Theorem 1. *If triangles ABC and $A'B'C'$ are orthologic with centers P, P' then the barycentric coordinates of P with respect to ABC are equal to the barycentric coordinates of P' with respect to $A'B'C'$.*

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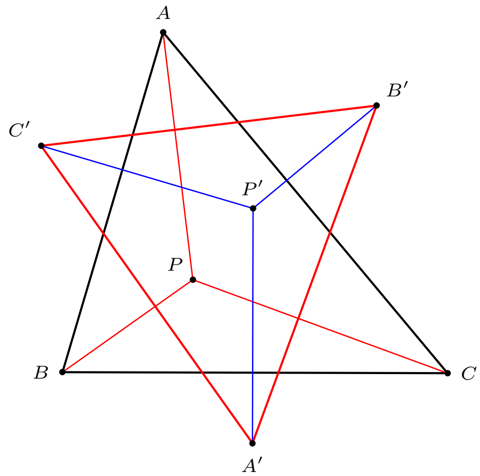

Figure 2

Proof. Since $A'P'$, $B'P'$, $C'P'$ are perpendicular to BC, CA, AB respectively, we have

$$\sin B'P'C' = \sin A, \quad \sin P'B'C' = \sin PAC, \quad \sin P'C'B' = \sin PAB.$$

Applying the law of sines to various triangles, we have

$$\begin{aligned}\frac{b}{P'B'} : \frac{c}{P'C'} =& \frac{1}{c \sin P'C'B'} : \frac{1}{b \sin P'B'C'} \\ =& \frac{1}{c \sin PAB} : \frac{1}{b \sin PAC} \\ =& \frac{1}{AP \cdot c \sin PAB} : \frac{1}{AP \cdot b \sin PAC} \\ =& \frac{1}{\text{area}(PAB)} : \frac{1}{\text{area}(PAC)} \\ =& \text{area}(PCA) : \text{area}(PAB).\end{aligned}$$

Similarly, $\frac{a}{P'A'} : \frac{b}{P'B'} = \text{area}(PBC) : \text{area}(PCA)$. It follows that the barycentric coordinates of P' with respect to triangle $A'B'C'$ are

$$\begin{aligned}&\text{area}(P'B'C') : \text{area}(P'C'A') : \text{area}(P'A'B') \\ =&(P'B')(P'C') \sin A : (P'C')(P'A') \sin B : (P'A')(P'B') \sin C \\ =&\frac{a}{P'A'} : \frac{b}{P'B'} : \frac{c}{P'C'} \\ =&\text{area}(PBC) : \text{area}(PCA) : \text{area}(PAB),\end{aligned}$$

the same as the barycentric coordinates of P with respect to triangle ABC. □

This property means that if P is the centroid of ABC then P' is also the centroid of $A'B'C'$.

2. The orthology center of $O_aO_bO_c$

We compute explicitly the coordinates (with respect to triangle ABC) of the orthology center Q of the triangle of circumcenters $O_aO_bO_c$. See Figure 3. Let $P=(x:y:z)$ and $Q=(u:v:w)$ in homogeneous barycentric coordinates. then $BC'=\frac{cx}{x+y}$, $CB'=\frac{bx}{x+z}$. In the notations of John H. Conway, the pedal A^* of O_a on BC has homogeneous barycentric coordinates $(0:uS_C+a^2v:uS_B+a^2w)$. See, for example, [6, pp.32, 49].

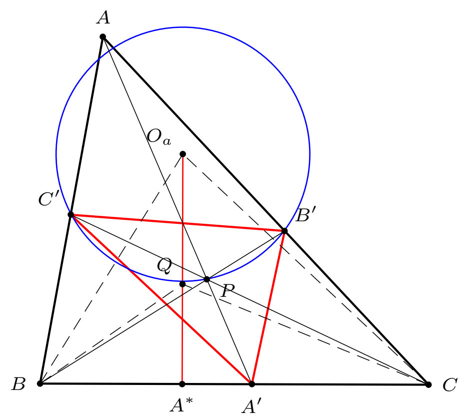

Figure 3

Note that $BA^*=\frac{uS_B+a^2w}{(u+v+w)a}$ and $A^*C=\frac{uS_C+a^2v}{(u+v+w)a}$. Also, by Stewart's theorem,

$$BB'^2=\frac{c^2x^2+a^2z^2+(c^2+a^2-b^2)xz}{(x+z)^2},$$
$$CC'^2=\frac{b^2x^2+a^2y^2+(a^2+b^2-c^2)xy}{(x+y)^2}.$$

Hence, if ρ is the circumradius of $PB'C'$, then

$$\begin{aligned}
&a(BA^*-A^*C)\\
=&(BA^*+A^*C)(BA^*-A^*C)\\
=&(BA^*)^2-(A^*C)^2\\
=&(O_aB)^2-(O_aA^*)^2-(O_aC)^2+(O_aA^*)^2\\
=&(O_aB)^2-\rho^2-(O_aC)^2+\rho^2\\
=&BP\cdot BB'-CP\cdot CC'\\
=&\frac{c^2x^2+a^2z^2+(c^2+a^2-b^2)xz}{(x+z)(x+y+z)}-\frac{b^2x^2+a^2y^2+(a^2+b^2-c^2)xy}{(x+y)(x+y+z)}\\
=&-\frac{a^2(y-z)(x+y)(x+z)+b^2x(x+y)(x+2z)-c^2x(x+z)(x+2y)}{(x+y)(x+z)(x+y+z)}
\end{aligned}$$

since the powers of B and C with respect to the circle of $PB'C'$ are $BB' \cdot BP = (O_aB)^2 - \rho^2$ and $CC' \cdot CP = (O_aC)^2 - \rho^2$ respectively. In other words,

$$\frac{(c^2-b^2)u - a^2(v-w)}{u+v+w} = -\frac{a^2(y-z)(x+y)(x+z) + b^2x(x+y)(x+2z) - c^2x(x+z)(x+2y)}{(x+y)(x+z)(x+y+z)},$$

or

$$\begin{aligned}&(a^2(y-z)(x+y)(x+z) - b^2(x+y)(xy+yz+z^2) + c^2(x+z)(y^2+xz+yz))u\\ -&(a^2(x+y)(x+z)(x+2z) - b^2x(x+y)(x+2z) + c^2x(x+z)(x+2y))v\\ +&(a^2(x+y)(x+z)(x+2y) + b^2x(x+y)(x+2z) - c^2x(x+z)(x+2y))w = 0.\end{aligned}$$

By replacing x, y, z by y, z, x and u, v, w by v,w, u, we obtain another linear relation in u, v, w. From these we have $u : v : w$ given by

$$\begin{aligned}u =&(x^2-z^2)y^2S_{BB} + (x^2-y^2)z^2S_{CC} - x(2x+y)(x+z)(y+z)S_{AB}\\ &- x(2x+z)(x+y)(y+z)S_{CA} - 2(x+y)(x+z)(xy+yz+zx)S_{BC}.\end{aligned}$$

and v obtained from u by replacing x, y, z, S_A, S_B, S_C by v, w, u, S_B, S_C, S_A respectively, and w from v by the same replacements.

3. Examples

3.1. *The centroid.* For $P = G$,

$$\begin{aligned}O_a =&(5S_A(S_B+S_C) + 2(S_{BB} + 5S_{BC} + S_{CC})\\ &: 3S_{AB} + 4S_{AC} + S_{BC} - 2S_{CC}\\ &: 3S_{AC} + 4S_{AB} + S_{BC} - 2S_{BB}).\end{aligned}$$

Similarly, we write down the coordinates of O_b and O_c. The perpendiculars from O_a to BC, from O_b to CA, and from O_c to AB have equations

$$\begin{array}{rcrcrcl}(S_B-S_C)x & - & (3S_B+S_C)y & + & (S_B+3S_C)z & = & 0,\\ (S_C+3S_A)x & + & (S_C-S_A)y & - & (3S_C+S_A)z & = & 0,\\ -(3S_A+S_B)x & + & (S_A+3S_B)y & + & (S_A-S_B)z & = & 0.\end{array}$$

These three lines intersect at the nine-point center

$$X_5 = (S_{CA} + S_{AB} + 2S_{BC} : S_{AB} + S_{BC} + 2S_{CA} : S_{BC} + S_{CA} + 2S_{AB}),$$

which is the orthology center of $O_aO_bO_c$.

3.2. *The orthocenter.* If P is the orthocenter, the circumcenters O_a, O_b, O_c are simply the midpoints of the segments AP, BP, CP respectively. In this case, $Q = H$.

3.3. *The Steiner point.* If P is the Steiner point $\left(\frac{1}{S_B-S_C} : \frac{1}{S_C-S_A} : \frac{1}{S_A-S_B}\right)$, the perpendiculars from the circumcenters to the sidelines are

$$\begin{array}{rcccccl} (S_B-S_C)x & - & S_Cy & + & S_Bz & = & 0, \\ S_Cx & + & (S_C-S_A)y & - & S_Az & = & 0, \\ -S_Bx & + & S_Ay & + & (S_A-S_B)z & = & 0. \end{array}$$

These lines intersect at the deLongchamps point

$$X_{20} = (S_{CA}+S_{AB}-S_{BC} : S_{AB}+S_{BC}-S_{CA} : S_{BC}+S_{CA}-S_{AB}).$$

3.4. X_{671}. The point $P = X_{671} = \left(\frac{1}{S_B+S_C-2S_A} : \frac{1}{S_C+S_A-2S_B} : \frac{1}{S_A+S_B-2S_C}\right)$ is the antipode of the Steiner point on the Steiner circum-ellipse. It is also on the Kiepert hyperbola, with Kiepert parameter $-\text{arccot}(\frac{1}{3}\cot\omega)$, where ω is the Brocard angle. In this case, the circumcenters are on the altitudes. This means that $Q = H$.

3.5. *An antipodal pair on the circumcircle.* The point X_{925} is the second intersection of the circumcircle with the line joining the deLongchamps point X_{20} to X_{74}, the isogonal conjugate of the Euler infinity point. It has coordinates

$$\left(\frac{1}{(S_B-S_C)(S^2-S_{AA})} : \frac{1}{(S_C-S_A)(S^2-S_{BB})} : \frac{1}{(S_A-S_B)(S^2-S_{CC})}\right).$$

For $P = X_{925}$, the orthology Q of $O_aO_bO_c$ is the point X_{68}, [1] which lies on the same line joining X_{20} to X_{74}.

The antipode of X_{925} is the point

$$X_{1300} = \left(\frac{1}{S_A((S_{AA}-S_{BC})(S_B+S_C)-S_A(S_B-S_C)^2)} : \cdots : \cdots\right).$$

It is the second intersection of the circumcircle with the line joining the orthocenter to the Euler reflection point [2] $X_{110} = \left(\frac{S_B+S_C}{S_B-S_C} : \frac{S_C+S_A}{S_C-S_A} : \frac{S_A+S_B}{S_A-S_B}\right)$. For $P = X_{1300}$, the orthology center Q of $O_aO_bO_c$ has first barycentric coordinate

$$\frac{S_{AA}(S_{BB}+S_{CC})(S_A(S_B+S_C)-(S_{BB}+S_{CC}))+S_{BC}(S_B-S_C)^2(S_{AA}-2S_A(S_B+S_C)-S_{BC}))}{S_A((S_B+S_C)(S_{AA}-S_{BC})-S_A(S_B-S_C)^2)}.$$

In this case, $O_aO_bO_c$ is also perspective to ABC at

$$X_{254} = \left(\frac{1}{S_A((S_{AA}-S_{BC})(S_B+S_C)-S_A(S_{BB}+S_{CC}))} : \cdots : \cdots\right).$$

By a theorem of Mitrea and Mitrea [5], this perspector lies on the line PQ.

[1] X_{68} is the perspector of the reflections of the orthic triangle in the nine-point center.

[2] The Euler reflection point is the intersection of the reflections of the Euler lines in the sidelines of triangle ABC.

3.6. More generally, for a generic point P on the circumcircle with coordinates $\left(\frac{S_B+S_C}{(S_A+t)(S_B-S_C)} : \cdots : \cdots\right)$, the center of orthology of $O_aO_bO_c$ is the point

$$\left(\frac{(S_B+S_C)(F(S_A,S_B,S_C)+G(S_A,S_B,S_C)t)}{S_A+t} : \cdots : \cdots\right),$$

where

$$\begin{aligned} F(S_A,S_B,S_C) =& S_{AA}(S_{BB}+S_{CC})(S_A+S_B+S_C)+S_{AABC}(S_B+S_C) \\ & - S_{BB}S_{CC}(2S_A+S_B+S_C), \\ G(S_A,S_B,S_C) =& 2(S_{AA}(S_{BB}+S_{BC}+S_{CC})-S_{BB}S_{CC}). \end{aligned}$$

Proposition 2. *If P lies on the circumcircle, the line joining P to Q always passes through the deLongchamps point X_{20}.*

Proof. The equation of the line PQ is

$$\begin{aligned} \sum_{\text{cyclic}} & (S_B-S_C)(S_A+t)(S_A^3(S_B-S_C)^2 \\ & + (S_B+S_C+2t)(S_{AA}(S_{BB}-S_{BC}+S_{CC})-S_{BB}S_{CC})x = 0. \end{aligned}$$

□

3.7. *Some further examples.* We conclude with a few more examples of P with relative simple coordinates for Q, the orthology center of $O_aO_bO_c$.

P	first barycentric coordinate of Q
X_7	$4a^3+a^2(b+c)-2a(b-c)^2-3(b+c)(b-c)^2$
X_8	$4a^4-5a^3(b+c)-a^2(b^2-10bc+c^2)+5a(b-c)^2(b+c)-3(b^2-c^2)^2$
X_{69}	$3a^6-4a^4(b^2+c^2)+a^2(3b^4+2b^2c^2+3c^4)-2(b^2-c^2)^2(b^2+c^2)$
X_{80}	$\frac{4a^3-3a^2(b+c)-2a(2b^2-5bc+2c^2)+3(b-c)^2(b+c)}{(b^2+c^2-a^2-bc)}$

In each of the cases $P = X_7$ and X_{80}, the triangle $O_aO_bO_c$ is also perspective to ABC at the incenter.

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Eric Danneels: Hubert d'Ydewallestraat 26, 8730 Beernem, Belgium
E-mail address: eric.danneels@pandora.be

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: ndergiades@yahoo.gr