The beautiful geometric theorem of Van Aubel

Article · January 2011

1 author:

Yutaka Nishiyama
Osaka University of Economics

International Journal of Pure and Applied Mathematics
Volume 66 No. 1 2011, 71-80

THE BEAUTIFUL GEOMETRIC THEOREM OF VAN AUBEL

Yutaka Nishiyama
Department of Business Information
Faculty of Information Management
Osaka University of Economics
2, Osumi, Higashiyodogawa, Osaka, 533-8533, JAPAN
e-mail: nishiyama@osaka-ue.ac.jp

Abstract: There is much beauty to be found among the theorems of geometry. This article introduces Van Aubel's Theorem (1878), and provides proofs using (i) vectors and complex numbers and (ii) elementary geometry. The beauty of this theorem lies both in the theorem itself and also in its proofs. The appearance of geometry problems in university entrance examinations has declined dramatically since the adoption of mark sheets. Geometry is one of mathematics' great resources, and the author suggests the revival of geometry problems in the mathematics curriculum.

AMS Subject Classification: 51M04, 97G50, 97I80
Key Words: Van Aubel's Theorem, complex number, vector, elementary geometry

1. From a Certain Geometry Problem

The following problem appeared in a certain bulletins of mathematics which is delivered periodically by post. Consider an arbitrary quadrilateral $ABCD$. A square is drawn against each edge of the quadrilateral. The problem is to prove that the lines PR and QS which connect the center points P, Q, R, S of these squares have equal length and that they intersect at right angles (Figure 1). When I came across this problem, it struck me as a beautiful theorem. Just what makes it so would be a divergent discussion, but I imagine that any lover of mathematics would also find it unconditionally beautiful.

Received: October 27, 2010

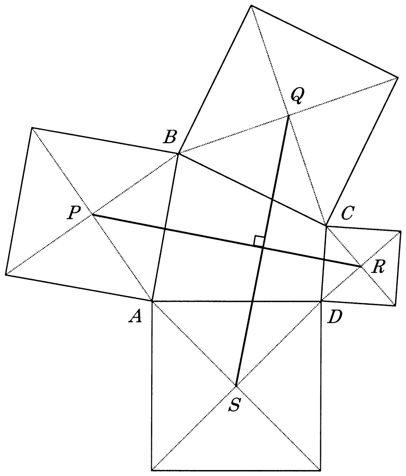

Figure 1

For me, the beauty in this problem lies not merely in the fact that for no matter what quadrilateral, the length of the line segments is equal and their intersection is at right angles, but also in the brilliance of the proof. Let us look at both below [2].

2. Proof Using Complex Numbers

For the quadrilateral $ABCD$, take the vertex A as an origin O. Next, represent the vector AB as the complex number $2a$, and likewise BC as $2b$, CD as $2c$, and DA as $2d$. The coefficient of 2 associated with each complex number is included for the sake of arithmetic convenience. The vector AP is represented by the complex number p, and likewise AQ by q, AR by r, and AS by s.

Since the quadrilateral $ABCD$ is closed, vector arithmetic yields $2a + 2b + 2c + 2d = 0$, i.e.

$$a + b + c + d = 0.$$

The proof is based on this constraint.

The point P may be reached from point A by proceeding half of the way from point A to point B, turning by 90 degrees, and then proceeding again by the same distance. The complex number p is therefore

$$p = a + ia = (1 + i)a.$$

Here i represents the imaginary unit such that $i^2 = -1$. Complex numbers may also be represented in polar form as (r, θ), in which case

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{\frac{\pi}{2}i},$$

so that multiplying a by i may be interpreted as multiplying it by a complex number with radius $r = 1$ and angle $\theta = \pi/2$, which thus yields no scaling of magnitude and is purely a rotation.

Likewise, the complex numbers q, r and s may be represented as follows:

$$q = 2a + (1 + i)b,$$

$$r = 2a + 2b + (1 + i)c,$$

$$s = 2a + 2b + 2c + (1 + i)d.$$

Denoting the vector from the point Q to the point S by A, and the vector from the point P to the point R by B, the following may be written since A is $s - q$ and B is $r - p$:

$$A = s - q = (b + 2c + d) + i(d - b),$$

$$B = r - p = (a + 2b + c) + i(c - a).$$

It must be proven that the line segments QS and PR are of equal length and are mutually perpendicular, that is to say, the relationship between the complex numbers A and B must satisfy

$$B = iA.$$

Alternatively, multiplying both sides of this equation by i and rearranging yields

$$A + iB = 0,$$

and it is thus sufficient to prove that this equation holds. Performing the actual calculation yields the following.

$$\begin{aligned} A + iB &= (b + 2c + d - c + a) + i(d - b + a + 2b + c) \\ &= (a + b + c + d) + i(a + b + c + d) = 0. \end{aligned}$$

This is really a brilliant proof. But somehow it is not enough. It remains unresolved whether or not the proof can only be made using complex numbers. The answer may be found below. Figure 3 is presented in [2] as a preliminary step for the proof.

Consider the triangle ABC formed by the edges AB and BC from the quadrilateral $ABCD$ shown in Figure 1. Squares are constructed on the outer

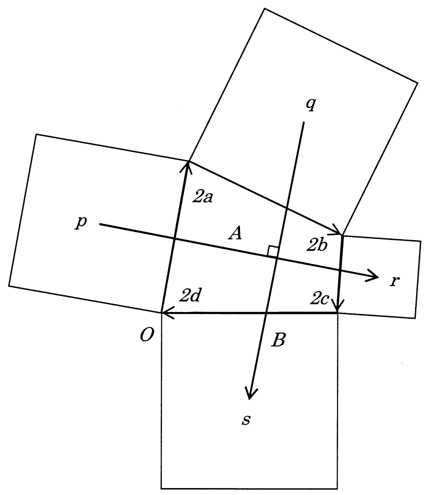

Figure 2

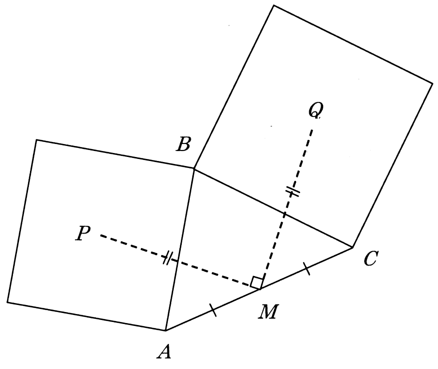

Figure 3

side of edges AB and BC, and their center points are denoted by P and Q respectively. Denoting the mid-point of the remaining edge CA by M, and constructing line segments PM and QM connecting M to P and Q respectively, the relationship between PM and QM is such that $|PM| = |QM|, PM \perp QM$. In order to prove this, complex numbers may be used in a similar manner to Figure 2, but let us think about the geometrical meaning.

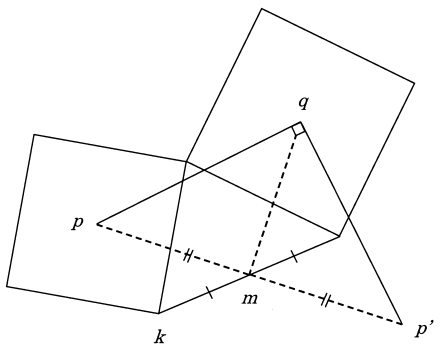

Figure 4

Translations and rotations may be investigated by using complex numbers. Denoting a rotation of $\pi/2$ about a point p by $R_p^{\pi/2}$, a rotation of $\pi/2$ about a point q by $R_q^{\pi/2}$, and a rotation of π about a point m by R_m^{π}, the composition of the these three rotations may be denoted by M such that

$$M = R_m^{\pi} \circ R_q^{\pi/2} \circ R_p^{\pi/2}.$$

Since $\pi/2 + \pi/2 + \pi = 2\pi$ the total rotation angle is an integer multiple of 2π, so for a certain point v, the transformation M is equivalent to a translation T_v.

Regarding this translation, for a certain point k, $M(k) = k$ so M is a 0 translation, that is to say, it is the identity transformation. Hence,

$$R_q^{\pi/2} \circ R_p^{\pi/2} = (R_m^{\pi})^{-1} \circ M = R_m^{\pi}.$$

If p' is defined such that $p' = R_m^{\pi}(p)$, then m is the mid-point of pp'. On the other hand, since

$$p' = (R_q^{\pi/2} \circ R_p^{\pi/2})(p) = R_q^{\pi/2}(p)$$

p' is the point p rotated by $\pi/2$ about the point q. Therefore pqp' forms a right-angled isosceles triangle, so pm and qm are perpendicular and have equal length (Figure 4).

3. Proof Using Elementary Geometry

This is sufficient as a proof, but I wondered if it is possible to construct a proof without using complex numbers. First, let us think about the proof using

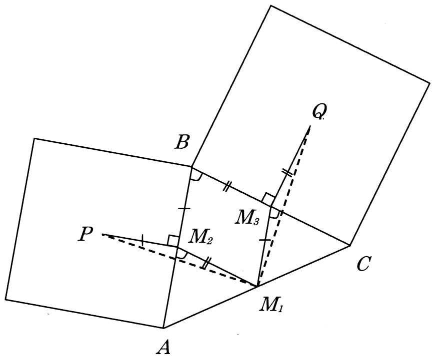

Figure 5

Figure 3 (see Figure 5).

Denote the mid-point of edge AC by M_1, the mid-point of edge AB by M_2, and of BC by M_3. Now $|M_2P| = |M_2B|, M_2P \perp M_2B$, $|M_3Q| = |M_3B|$ and $M_3Q \perp M_3B$. If M_2B is translated to M_1M_3, and M_3B is translated to M_1M_2 then $\triangle PM_1M_2$ and $\triangle QM_1M_3$ have two edges of equal length and an internal angle of 90 degrees plus $\angle B$, and are hence identical. Thus since

$$PM_2 \perp M_1M_3, \quad M_1M_2 \perp QM_3,$$

the remaining edges PM_1 and QM_1 have equal length and are perpendicular, see [1]. In terms of complex numbers, rotating $\triangle PM_1M_2$ by $\pi/2$ about the point M_2, and then translating it in the direction from M_2 to M_3 yields $\triangle QM_1M_3$.

Figure 1 may be proven using the result from Figure 3 twice. For an arbitrary quadrilateral $ABCD$, draw the line AC and denote its mid-point by M. Then for the squares P and Q,

$$|PM| = |QM|, \quad PM \perp QM$$

and for the squares R and S,

$$|RM| = |SM|, \quad RM \perp SM.$$

Let us think about $\triangle PMR$ and $\triangle QMS$. Since the two corresponding edges are equal, $|PM| = |QM|, |RM| = |SM|$, and they both have internal angles of $\angle QMR$ plus a right-angle (so these internal angles are equal), we know that

$$\triangle PMR \equiv \triangle QMS.$$

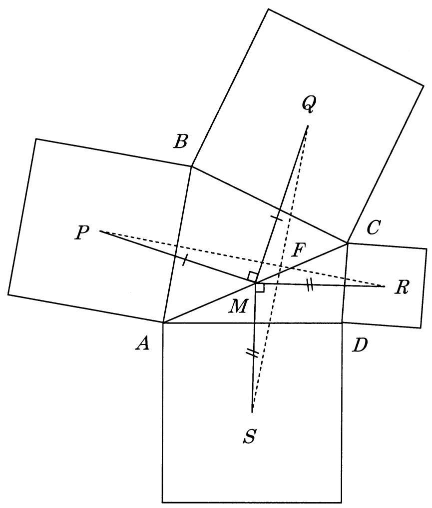

Figure 6

Also, since they have been rotated by 90 degrees about the point M,

$$|PR| = |QS|, \quad PR \perp QS.$$

The intersection of PR with QS, denoted F, differs in general from M.

Now, PR and QS have equal length and are perpendicular, but by what ratio does their intersection point F divide the line segments? In order to find out, the center points of each square may be connected to form PQ, QR, RS and SP, which also yields the right angled triangles $\triangle FQP$, $\triangle FRQ$, $\triangle FSR$ and $\triangle FPS$. The enclosing circles with diameters PQ, QR, RS and SP all intersect at point F. The lengths of the chords PF, QF, RF and SF formed by adjacent circles determine the ratio relating the line segments PR and QS (Figure 7). If the quadrilateral $ABCD$ is a square, rhombus, rectangle, or parallelogram then the quadrilateral $PQRS$ is a square and PR and QS divide each other into two equal halves.

4. Van Aubel's Theorem

It took me about two weeks to fully grasp the problem described in this article. To begin with, I wondered if the proposition was possible and drew myself di-

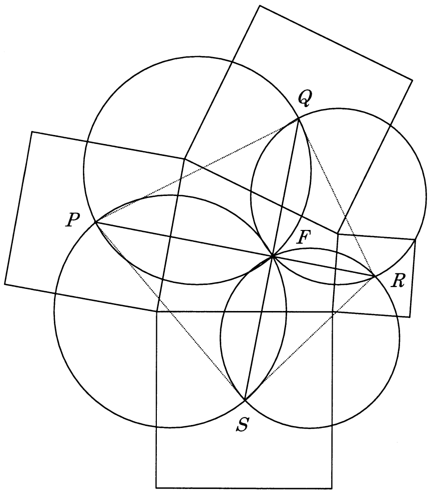

Figure 7

agrams to test various quadrilaterals. For squares, rhombuses, rectangles, and parallelograms, the line segments were clearly of equal length and perpendicular. For an isosceles trapezium, the line segments were clearly perpendicular but it was difficult to prove that they had equal length. When it came to a general trapezium, I was completely stumped.

Next, I input the coordinates of the vertices of the quadrilateral $ABCD$ into a computer to investigate them. I attempted to find the coordinates of the intersection point using straight line equations but it was difficult to expand the formula. It occurred to me that while this effort confirms the result of the theorem, such a verification does not constitute a proof and I decided to stop.

On the other hand, I thought it would be useful if there were some software available for constructing a figure showing PR and QS for an arbitrary quadrilateral $ABCD$. When I searched in Internet, I found exactly what I had been imagining. It was created as a small program known as a *Java Applet*, and discovering that it was even available for purchase, it made me aware that times have changed. Dragging each vertex of the quadrilateral $ABCD$, each of the corresponding squares change in response, as do the line segments PR and QS. It is possible to confirm that no matter how the vertices are moved, the two line segments remain perpendicular and retain an equal length.

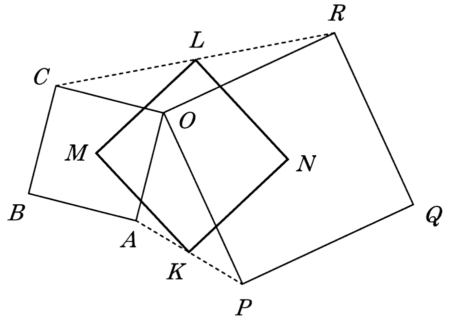

Figure 8

When I came across this geometry problem it struck me as novel, because it is not often presented in Japan. I therefore delved into its roots. It is introduced in the 1971 edition of the "Comprehensive Dictionary of Geometry, Volume 1", on page 296 as problem number 387, and two proofs – one using elementary geometry and the other using complex numbers – are included, see [1]. Problem number 386 is Morley's Theorem (the three points of intersection of the adjacent angle trisectors of any triangle form an equilateral triangle). It is classified as a "famous theorem", but it is odd that despite having a name it is not that well known.

On the other hand, one frequently comes across websites in Europe and the US which present this problem. There are related examples such as that shown in Figure 8. For example, the squares $ABCO$ and $PORQ$ are connected at the point O, and the line segments AP and CR which connect vertices of each square have been drawn. The mid-points of these lines denoted by K and L, and the centers of the squares denoted by M and N have been connected to form $MKNL$, which is a square. This can be proven by applying Figure 3 or Figure 5 twice.

Again, for an arbitrary triangle ABC, if each of its edges is taken as the edge of an equilateral triangle on the outer side of the edge, then connecting the barycenters of these equilateral triangles also forms an equilateral triangle. This is known as Napoleon's Theorem.

This is truly a beautiful theorem. Investigating the roots of these theorems leads me back to Van Aubel's work, see [3] of 1878. He is not a very well-known mathematician in Japan, but let us consider this beautiful result to be Van Aubel's Theorem.

It's been a long time since planar geometry was removed from high-school mathematics text books. One of the reasons it was removed lies in the reformation of university entrance examinations. The responses to university entrance examinations were formerly written, but were subsequently replaced by a mark sheet format. This made it difficult to include geometry questions. Since the number of pupils has begun to decrease due to the aging population and decreasing birth rate, perhaps examinations should once again adopt written responses and boldly revive geometry?

References

[1] Shiko Iwata, Kikagaku Daijiten, *Comprehensive Dictionary of Geometry*, **1**, Tokyo, Maki Shoten (1971), 123-296 (2003), In Japanese.

[2] Tristan Needham, *Visual Complex Analysis*, Oxford University Press (1997).

[3] M.H. Van Aubel, Note concernant les centres des carrés construits sur les côtés d'un polygon quelconque, *Nouv. Corresp. Math.*, **4** (1878), 40-44, In French.