BEBERA PEMBUKTIAN BILANGAN IRRASIONAL

$$\pi \approx 3.14159265358979323846264338327950288419716939937510...$$

$$\sqrt{2} \approx 1.41421356237309504880168872420969807856967187537694...$$

$$\sqrt{3} \approx 1.73205080756887729352744634150587236694280525381038...$$

BAGAIMANA DENGAN $\sqrt{2}$

Proposition 1. The square root of 2 is irrational.

We prove that $\sqrt{2}$ is irrational.

Assume to the contrary that $\sqrt{2}$ is rational, $\sqrt{2} = \frac{p}{q}$,

where p and q are integers and $q \neq 0$.

Moreover, let $\underline{p \text{ and } q \text{ have no common divisor} > 1}$.

Then

$$2 = \frac{p^2}{q^2} \quad \Rightarrow \quad 2q^2 = p^2. \tag{1}$$

Since $2q^2$ is even, it follows that p^2 is even.

Then <u>p is also even</u>

(in fact, if p is odd, then p^2 is odd).

This means that there exists $k \in \mathbb{Z}$ such that

$$p = 2k. \tag{2}$$

Substituting (2) into (1), we get

$$2q^2 = (2k)^2 \Rightarrow \quad 2q^2 = 4k^2 \quad \Rightarrow \quad q^2 = 2k^2.$$

Since $2k^2$ is even,

it follows that q^2 is even.

Then <u>q is also even.</u>

This is a contradiction. ■

We prove that $\sqrt[3]{4}$ is irrational.

Assume to the contrary that $\sqrt[3]{4}$ is rational, that is

$$\sqrt[3]{4} = \frac{p}{q},$$

where p and q are integers and $q \neq 0$.

Moreover, let p and q have no common divisor > 1.

Then

$$4 = \frac{p^3}{q^3} \quad \Rightarrow \quad 4q^3 = p^3. \tag{1}$$

Since $4q^3$ is even, it follows that p^3 is even.

Then $\underline{p \text{ is also even}}$

(in fact, if p is odd, then p^3 is odd).

This means

that there exists $k \in \mathbb{Z}$ such that

$$p = 2k. \tag{2}$$

Substituting (2) into (1), we get

$$4q^3 = (2k)^3 \quad \Rightarrow \quad 4q^3 = 8k^3 \quad \Rightarrow \quad q^3 = 2k^3.$$

Since $2k^3$ is even, it follows that q^3 is even. Then q is also even.

This is a contradiction. ■

??????$\sqrt{6}$

We prove that $\sqrt{6}$ is irrational.

Assume to the contrary that $\sqrt{6}$ is rational, that is

$$\sqrt{6} = \frac{p}{q},$$

where p and q are integers and $q \neq 0$.

Moreover, let $\underline{p \text{ and } q \text{ have no common divisor} > 1}$.

Then

$$6 = \frac{p^2}{q^2} \quad \Rightarrow \quad 6q^2 = p^2. \tag{1}$$

Since $6q^2$ is even, it follows that p^2 is even.

Then <u>p is also even</u> (in fact, if p is odd, then p^2 is odd).

This means that there exists $k \in \mathbb{Z}$ such that

$$p = 2k. \tag{2}$$

Substituting (2) into (1), we get

$$6q^2 = (2k)^2 \quad \Rightarrow \quad 6q^2 = 4k^2 \quad \Rightarrow \quad 3q^2 = 2k^2.$$

Since $2k^2$ is even, it follows that $3q^2$ is even.

Then <u>q is also even</u> (in fact, if q is odd, then $3q^2$ is odd).

contradiction. ■

We prove that $\frac{1}{3}\sqrt{2}+5$ is irrational.

Assume to the contrary that $\frac{1}{3}\sqrt{2}+5$ is rational,

that is $$\frac{1}{3}\sqrt{2}+5=\frac{p}{q},$$

where p and q are integers and $q \neq 0$.

Then $$\sqrt{2}=\frac{3(p-5q)}{q}.$$

Since $\sqrt{2}$ is irrational and $\frac{3(p-5q)}{q}$ is rational,

we obtain a contradiction.

We prove that $\log_5 2$ is irrational.

Assume to the contrary that $\log_5 2$ is rational, that is

$$\log_5 2 = \frac{p}{q},$$

where p and q are integers and $q \neq 0$. Then

$$5^{p/q} = 2 \quad \Rightarrow \quad 5^p = 2^q.$$

Since 5^p is odd and 2^q is even, we obtain a contradiction. ■

We prove that $\sqrt{2}+\sqrt{3}$ is irrational.

Assume to the contrary that $\sqrt{2}+\sqrt{3}$ is rational,

that is
$$\sqrt{2}+\sqrt{3}=\frac{p}{q},$$

where p and q are integers and $q \neq 0$. Then

$$\left(\sqrt{2}+\sqrt{3}\right)^2=\frac{p^2}{q^2} \quad \Rightarrow \quad 2+2\sqrt{2}\sqrt{3}+3=\frac{p^2}{q^2}$$

$$\Rightarrow \quad 5+2\sqrt{6}=\frac{p^2}{q^2} \quad \Rightarrow \quad \sqrt{6}=\frac{p^2-5q^2}{2q^2}.$$

Since $\sqrt{6}$ is irrational and $\dfrac{p^2-5q^2}{2q^2}$ is rational,

we obtain a contradiction. ■

We prove that $\sqrt{2}+\sqrt[3]{3}$ is irrational.

Assume to the contrary that $\sqrt{2}+\sqrt[3]{3}$ is rational,

, that is

$$\sqrt{2}+\sqrt[3]{3}=\frac{p}{q},$$

where p and q are integers and $q \neq 0$.

It follows that

$$\sqrt[3]{3}=\frac{p}{q}-\sqrt{2},$$

hence

$$\sqrt[3]{3} = \frac{p}{q} - \sqrt{2},$$

$$\begin{aligned}
3 &= \left(\frac{p}{q} - \sqrt{2}\right)^3 \\
&= \frac{p^3}{q^3} - 3\frac{p^2}{q^2}\sqrt{2} + 3\frac{p}{q}\left(\sqrt{2}\right)^2 - \left(\sqrt{2}\right)^3 \\
&= \frac{p^3}{q^3} - 3\frac{p^2}{q^2}\sqrt{2} + 6\frac{p}{q} - 2\sqrt{2} \\
&= \frac{p^3}{q^3} + 6\frac{p}{q} - \sqrt{2}\left(3\frac{p^2}{q^2} + 2\right).
\end{aligned}$$

We can rewrite this as

$$\sqrt{2} = \frac{\dfrac{p^3}{q^3} + 6\dfrac{p}{q} - 3}{3\dfrac{p^2}{q^2} + 2} = \frac{p^3 + 6pq^2 - 3q^3}{3p^2q + 2q^3}.$$

Since $\sqrt{2}$ is irrational and $\dfrac{p^3 + 6pq^2 - 3q^3}{3p^2q + 2q^3}$ is rational,

we obtain a contradiction. ■

Proposition 2. *For any squarefree integer $n > 1$, $\sqrt{n}$ is irrational.*

What about the case of general n? Well, of course $\sqrt{n^2}$ is not only rational but is an integer, namely n. Moreover, an arbitrary positive integer n can be factored to get one of these two limiting cases: namely, any n can be uniquely decomposed as

$$n = sN^2,$$

where s is squarefree. (Prove it!) Since $\sqrt{sN^2} = N\sqrt{s}$, we have that $\sqrt{n}$ is rational iff $\sqrt{s}$ is rational; by the above result, this only occurs if $s = 1$. Thus:

Theorem 3. *For $n \in \mathbb{Z}^+$, $\sqrt{n}$ is rational iff $n = N^2$ is a perfect square.*

Another way of stating this result is that $\sqrt{n}$ is either an integer or is irrational.

What about cube roots and so forth? We can prove that $\sqrt[3]{2}$ is irrational using a similar argument: suppose $\sqrt[3]{2} = \frac{a}{b}$, with $\gcd(a, b) = 1$. Then we get

$$2b^3 = a^3,$$

so $2 \mid a^3$, thus $2 \mid a$. Put $a = 2A$, so $b^3 = 2^2 A^3$ and $2 \mid b^3$. Thus $2 \mid b$: contradiction.

Any integer can be written as the product of a cube-free integer[1] and a perfect cube; with this one can prove that the $\sqrt[3]{n}$ is irrational unless $n = N^3$. For the sake of variety, we prove the general result in a different way.

Theorem 4. *Let $k > 2$ be a positive integer. Then $\sqrt[k]{n}$ is irrational unless $n = N^k$ is a perfect kth power.*

Proof. Suppose n is not a perfect kth power. Then there exists some prime $p \mid n$ such that $\operatorname{ord}_p(n)$ is not divisible by k. Let us use this prime to get a contradiction:

$$\frac{a^k}{b^k} = n, \; a^k = nb^k.$$

Take ord_p of both sides:

$$k \operatorname{ord}_p(a) = \operatorname{ord}_p(a^k) = \operatorname{ord}_p(nb^k) = k \operatorname{ord}_p(b) + \operatorname{ord}_p(n),$$

so $\operatorname{ord}_p(n) = k(\operatorname{ord}_p(a) - \operatorname{ord}_p(b))$ and $k \mid \operatorname{ord}_p(n)$: contradiction. □

Bagaimana irrasional dari e dan π

SEKARANG BENTUK YANG LAIN

We prove that $\sin 1^\circ$ is irrational.

Assume to the contrary that $\sin 1^\circ$ is rational.

Then $\cos^2 1^\circ$ and $\cos 2^\circ$ are also rational,

since

$$\cos^2 1^\circ = 1 - \sin^2 1^\circ \quad \text{and} \quad \cos 2^\circ = \cos^2 1^\circ - \sin^2 1^\circ.$$

Similarly, $\cos 4^\circ$, $\cos 8^\circ$, $\cos 16^\circ$, and $\cos 32^\circ$ are rational, since

$$\cos 4^\circ = 2\cos^2 2^\circ - 1, \quad \cos 8^\circ = 2\cos^2 4^\circ - 1,$$

$$\cos 16^\circ = 2\cos^2 8^\circ - 1, \quad \cos 32^\circ = 2\cos^2 16^\circ - 1.$$

On the other hand, we have

$$\frac{\sqrt{3}}{2} = \cos 30^\circ = \cos(32^\circ - 2^\circ):$$

$$= \cos 32^\circ \cos 2^\circ + \sin 32^\circ \sin 2^\circ$$

$$= \cos 32^\circ \cos 2^\circ + 2\cos 16^\circ \sin 16^\circ \sin 2^\circ$$

$$= \cos 32^\circ \cos 2^\circ + 4\cos 16^\circ \cos 8^\circ \sin 8^\circ \sin 2^\circ$$

$$= \cos 32^\circ \cos 2^\circ + 8\cos 16^\circ \cos 8^\circ \cos 4^\circ \sin 4^\circ \sin 2^\circ$$

$$= \cos 32^\circ \cos 2^\circ + 16\cos 16^\circ \cos 8^\circ \cos 4^\circ \cos 2^\circ \sin^2 2^\circ$$

$$= \cos 32^\circ \cos 2^\circ + 64\cos 16^\circ \cos 8^\circ \cos 4^\circ \cos 2^\circ \cos^2 1^\circ \sin^2 1^\circ.$$

The right-hand side is rational. One can prove that $\frac{\sqrt{3}}{2}$ is irrational.

We obtain a contradiction. ■

We prove that $2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\ldots$ is irrational.

Assume to the contrary that this number is rational, that is

$$\frac{p}{q}=2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\ldots,$$

where p and q are integers and $q \neq 0$.

We multiply both sides by $qn!$ with $n > q$.

We get

$$pn! = qn!\left(2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\ldots\right)$$

$$pn! = qn!\left(2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\ldots\right)$$
$$= qn!\left(2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}\right)+qn!\left(\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\frac{1}{(n+3)!}+\ldots\right)$$
$$= qn!\left(2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}\right)+q\left(\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\ldots\right),$$

so

$$pn! - qn!\left(2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}\right)$$
$$= q\left(\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\ldots\right)$$

Note that $pn!$ and $qn!$

$$\left(2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}\right) \text{ are integer.}$$

If we prove that

$$q\left(\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\dots\right)<1,$$

we obtain a contradiction.

To this end we observe that

$$\frac{1}{(n+1)(n+2)}<\frac{1}{(n+1)^2},$$

$$\frac{1}{(n+1)(n+2)(n+3)}<\frac{1}{(n+1)^3},\dots.$$

By this and a formula of geometric progression we have

$$q\left(\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\dots\right)$$

$$< q\left(\frac{1}{n+1}+\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}+\dots\right)$$

$$= q\frac{1}{(n+1)\left(1-\frac{1}{n+1}\right)}$$

$$= q\frac{1}{n+1-\frac{n+1}{n+1}} = q\frac{1}{n+1-1} = \frac{q}{n},$$

which is < 1 by (1). ■